Randomized Convex Optimization

Panagiotis REPOUSKOS

Sampling Methods for Convex Optimization

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Joined work with:
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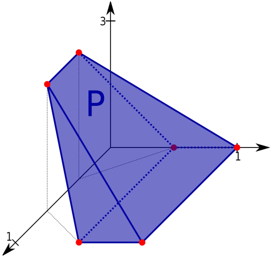
Section 1

Convex Optimization and Random Walks

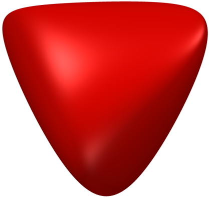
Linear Programming

$$\begin{aligned} \text{Minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$



Semidefinite Programming

$$\begin{aligned} \text{Minimize} \quad & C \cdot X \\ \text{subject to} \quad & A_i \cdot X \leq b_i \\ & X \succeq 0 \end{aligned}$$



[1] $C \cdot X := \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$

[2] The images were taken from Wikipedia

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Convex Optimization and Random Walks
Convex Optimization
Random Walks

Randomized Cutting Plane
Algorithm
Experiments

Simulated Annealing
Algorithm

Future Work

References

- We will use sampling to solve convex programs

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- We need to sample a convex body under some distribution

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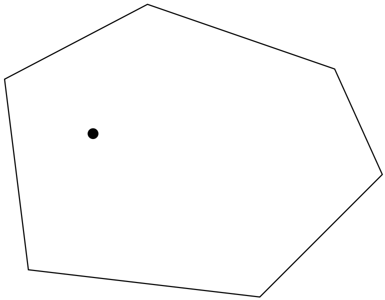
- We need to sample a convex body under some distribution
- Use a geometric random walk
- A geometric random walk is a Markov Chain
- We also need a boundary oracle

A Random Walk (Billiard Walk)

- Decide a distance H and repeat k times



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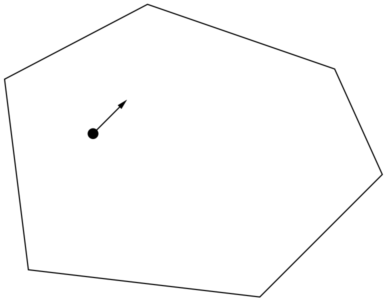

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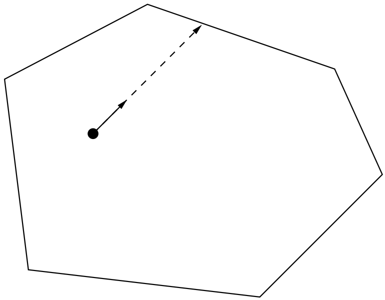

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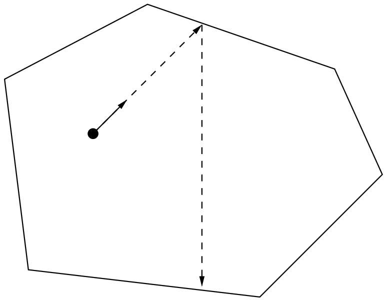

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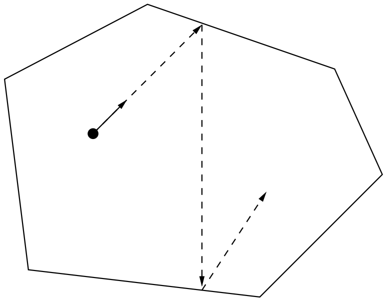

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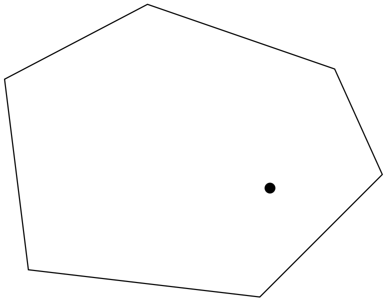
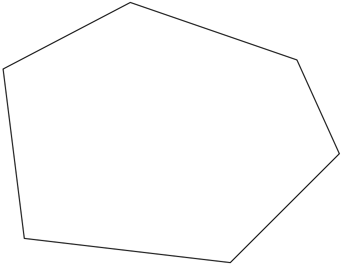


Section 2

Randomized Cutting Plane

An Algorithm for Approximation

- Input: convex body K, objective function c



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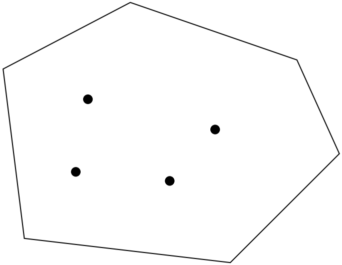

Algorithm

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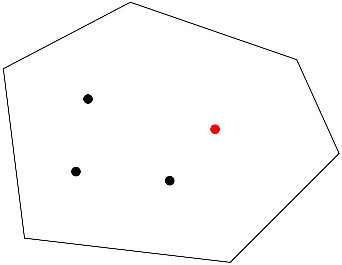
- Sample N points under the uniform distribution
- Find the point x minimizing the objective function



Algorithm

- Input: convex body K, objective function c
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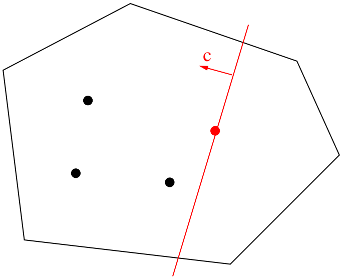
- Cut the convex body at x



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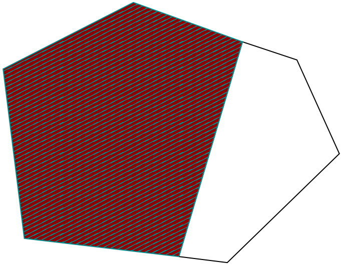

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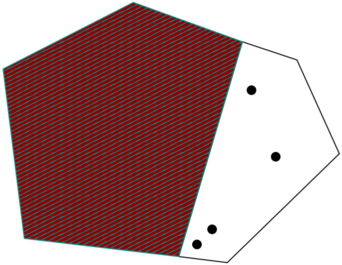

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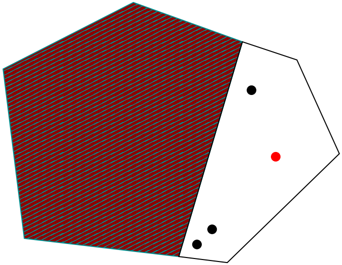

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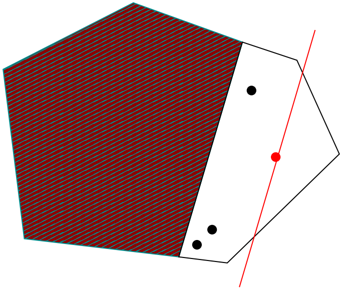

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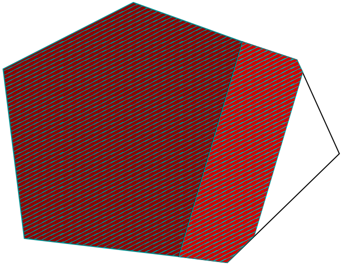

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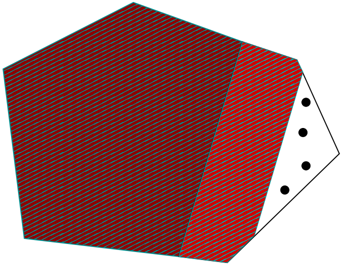


Expected Convergence [Dabbene, Shcherbakov, and Polyak]

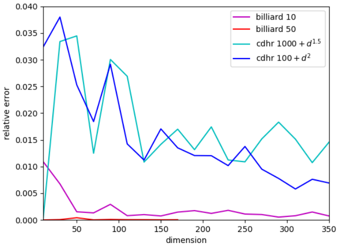
The expected number of steps k, to get a solution x_k s.t. $x_k - x^\star \leq \alpha$ is at most:

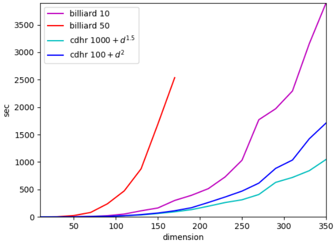
$$k = \lceil \frac{1}{\ln(N+1)} n \ln R/\alpha \rceil$$

where R is the diameter of the convex body in $\mathbb{R}^n$ and $N = \#$ samples.

Results for Polytopes

Our implementation using CDH&R[1] and Billiard Walk.





[1] Hit and Run with coordinate directions

[2] The polytopes tested are in n dimensions and have $4n$ facets.

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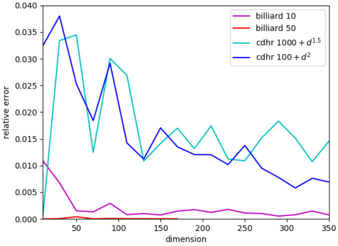

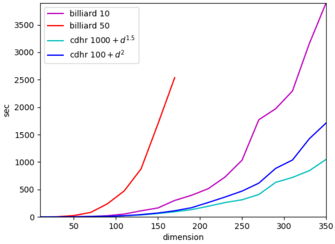

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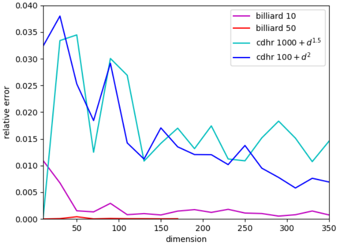
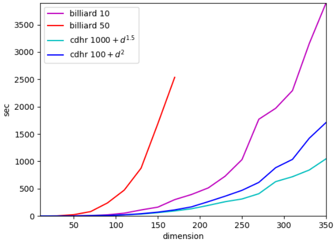


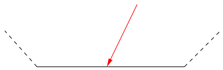


- Different random walks offer different benefits.
- Billiard offers precision, CDH&R offers speed.

[1] Hit and Run with coordinate directions

[2] The polytopes tested are in n dimensions and have $4n$ facets.

- The boundary oracle for polytopes entails linear equations.

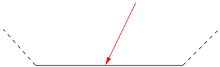


Linear Programming

$$\begin{aligned} \text{Minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Moving to Spectrahedra

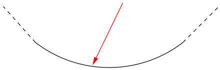
- The boundary oracle for polytopes entails linear equations.



Linear Programming

$$\begin{array}{ll} \text{Minimize} & c^\top x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

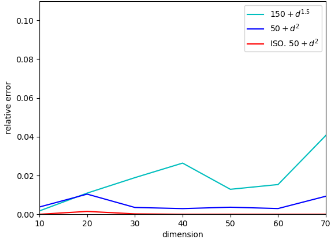
- The boundary oracle for spectrahedra entails eigenvalues (equivalently polynomial equations).

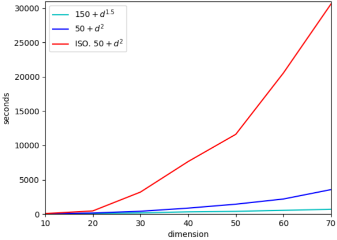


Semidefinite Programming

$$\begin{array}{ll} \text{Minimize} & C \cdot X \\ \text{subject to} & A_i \cdot X \leq b_i \\ & X \succeq 0 \end{array}$$

Using RDH&R[1] with / without implicit isotropization.



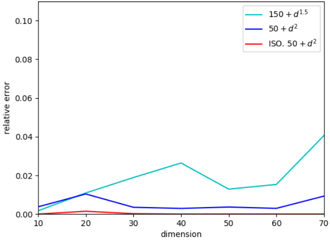


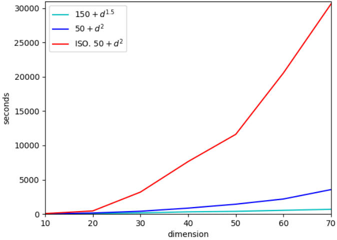
[1] Hit and Run with random directions

[2] The matrices in the experiments are of dimension 50

H&R on Spectrahedra

Using RDH&R[1] with / without implicit isotropization.





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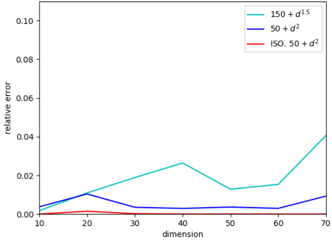

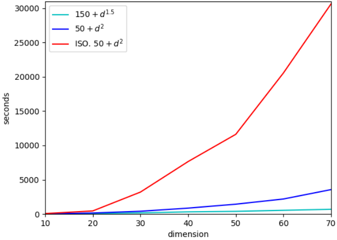

- This random walk doesn't scale well with dimensions
- We need better sampling for spectrahedra

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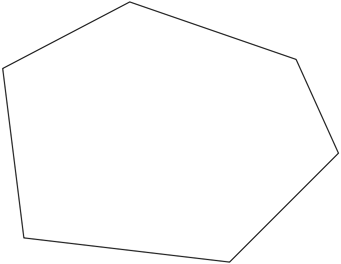
Section 3

Simulated Annealing

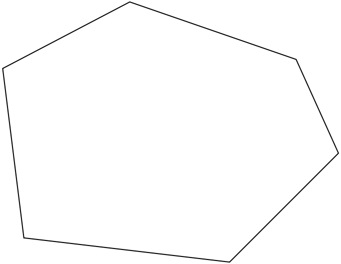
Beyond Uniform Distribution

Algorithm

- For $i = 1, 2, ..., I$

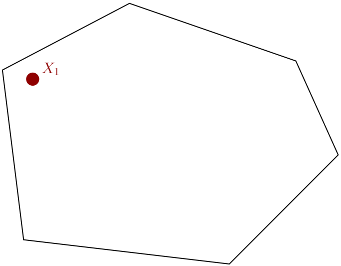


- For $i = 1, 2, \ldots, I$
  - set temperature $T_i = R\left(1 - \frac{1}{\sqrt{n}}\right)^i$



Algorithm

- For $i = 1, 2, ..., I$
  - set temperature $T_i = R\left(1 - \frac{1}{\sqrt{n}}\right)^i$
  - get a sample X_i using H&R with density $e^{-c \cdot x/T}$



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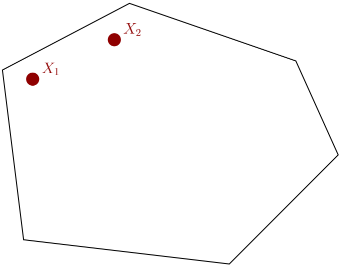

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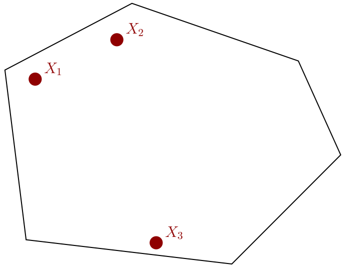

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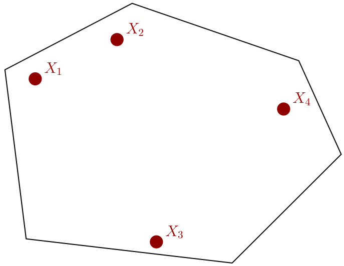

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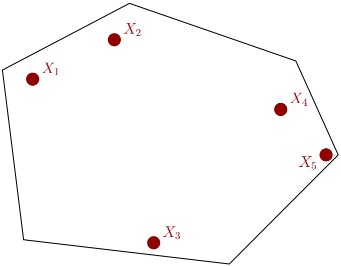

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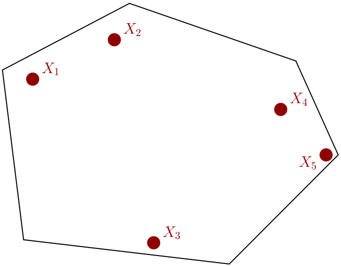


- Return X_I

Theorem [Kalai and Vempala]

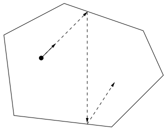
With probability $1-\delta$, $I = O(\sqrt{n}\log(Rn/\epsilon\delta))$ get a X_I s.t.

$$c \cdot X_I \leq \min_{x \in K} c \cdot x + \epsilon$$

Section 4

Future Work

Future Work



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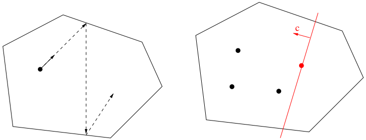

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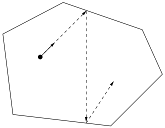

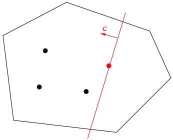

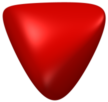

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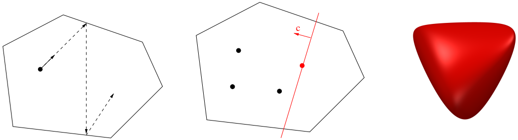

- Billiard Walk for spectrahedra - need to efficiently compute reflections

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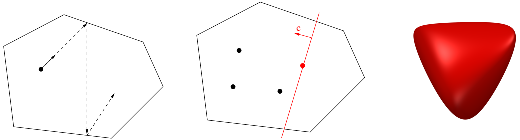
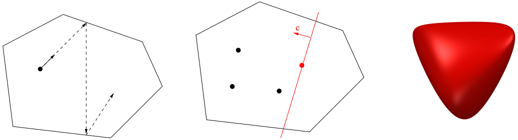


- Billiard Walk for spectrahedra - need to efficiently compute reflections
- Efficient boundary oracle for CDH&R in spectrahedra

Future Work



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- Efficient boundary oracle for CDH&R in spectrahedra
- Study behavior of simulated annealing

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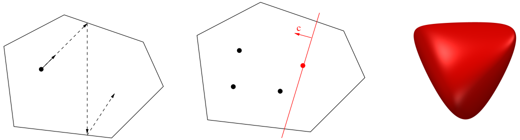

- Billiard Walk for spectrahedra - need to efficiently compute reflections
- Efficient boundary oracle for CDH&R in spectrahedra
- Study behavior of simulated annealing
- Try new random walks, including Hamiltonian Monte Carlo

R project for statistical computing

Google Summer of Code 2019

Project: Sampling Methods for Convex Optimization

Mentors: Vissarion Fisikopoulos
Elias Tsigaridas
Zafeirakis Zafeirakopoulos

[1] https://github.com/GeomScale/volume_approximation

References

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A. T. Kalai and S. Vempala. Simulated Annealing for Convex Optimization. *Mathematics of Operations Research*, 31(2):253–266, May 2006. ISSN 0364-765X, 1526-5471. doi: 10.1287/moor.1060.0194. URL http://pubsonline.informs.org/doi/abs/10.1287/moor.1060.0194.